AD-A032 922 NAPLES UNIV (ITALY) ISTITUTO DI AERODINAMICA F/G 12/1
APPLICATION OF FUNCTIONAL ANALYSIS IN FLUID-MECHANICS.(U)
JUL 76 L G NAPOLITANO AF-AFOSR-2889-76
UNCLASSIFIED IA-235 AFOSR-TR-76-1185 NL

1 OF 1
ADA032922

END
DATE
FILMED
1 -77

AFOSR - TR - 76 - 1185

GRANT AFOSR-76-2889

JULY, 1976

12

ADA032922

FINAL SCIENTIFIC REPORT

APPLICATION OF FUNCTIONAL ANALYSIS IN FLUID-MECHANICS

BY

L. G. NAPOLITANO

AERODYNAMICS INSTITUTE, UNIVERSITY OF NAPLES

I.A. REPORT No.235

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

18

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 19 AFOSR - TR - 76 - 1185 / 2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) 6 APPLICATION OF FUNCTIONAL ANALYSIS IN FLUID-MECHANICS.	9 5. TYPE OF REPORT & PERIOD COVERED FINAL rept. 1 July 1975 — 1 July 1976
	14 6. PERFORMING ORG. REPORT NUMBER IA-235
7. AUTHOR(s) 10 L.G. NAPOLITANO	15 8. CONTRACT OR GRANT NUMBER(s) ✓ AF-AFOSR-2889-76
9. PERFORMING ORGANIZATION NAME AND ADDRESS UNIVERSITY OF NAPLES INSTITUTE OF AERODYNAMICS, P.le V. TECCHIO, 80 80126 NAPLES, ITALY	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 681307 16 9781-01 17 01 61102F
11. CONTROLLING OFFICE NAME AND ADDRESS AIR FORCE OFFICE SCIENTIFIC RESEARCH/NA BLDG 410 BOLLING AIR FORCE BASE, D C 20332	11 12. REPORT DATE July 1976
	13. NUMBER OF PAGES 38
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) 12 43p.	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

VARIATIONAL FUNCTIONALS
SPLINE FUNCTIONS
INVISCID FLOW

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The report gives the appropriate references of papers originated under the subject Grant. In addition it presents in details the theory underlying an inverse functional approach for the derivation of variational formulations (classical, dual and hybrid) of continuum mechanics linear problems (Part I) and the theory of a new class of spline functions which are herein called "closed spline functions" (Part II).

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED 400 197 LB

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

July, 1976

FINAL SCIENTIFIC REPORT
GRANT AFOSR-76-2889

APPLICATION OF FUNCTIONAL ANALYSIS IN FLUID-MECHANICS

prepared by
L.G.NAPOLITANO
Aerodynamics Institute, University of Naples

This research has been sponsored in part by the Air Force Office of Scientific Research through the European Office of Aerospace Research OAR, United States Air Force, Under Grant No. AFOSR-76-2889.

T A B L E O F C O N T E N T

.- FOREWORD

PART I - THE FUNCTIONAL INVERSE APPROACH FOR THE VARIATIONAL FORMULATION OF BOUNDARY VALUE PROBLEMS

1.- INTRODUCTION

2.- FUNDAMENTALS OF THE INVERSE APPROACH p. 1

2.1 - The abstract Gauss formula

2.2 - Constraint and solutions spaces p. 3

2.3 - The associated boundary value problem for compatible elements p. 6

2.4 - Linear varieties and two-field variational functional p. 7

3.- VARIATIONAL FUNCTIONALS p. 8

4.- GENERALIZATIONS p. 13

CONCLUDING REMARKS p. 15

REFERENCES p. 16

PART II - THE CLOSED SPLINE FUNCTIONS

1.- INTRODUCTION p. 17

2.- THE ABSTRACT SPLINE FUNCTION THEORY p. 20

3.- THE CLOSED SPLINE-FUNCTIONS - EXISTENCE,UNIQUENESS AND CHARACTERIZATION p. 23

4.- THE CLOSED SPLINE FUNCTIONS - DEFINITION, CHARACTERIZATION AND PROPERTIES p. 30

CONCLUDING REMARKS p. 35

REFERENCES p. 36

0.- FOREWORD

The work done during the period covered by the present FINAL Scientific Report pertains to several topics related to the application of functional analysis in fluid-mechanics.

It has lead to the following papers :

1.- L.G. NAPOLITANO :"Functional Analysis Approach for the Derivation of Hybrid Variational Functionals".

This paper was presented at the IUTAM/IUM Symposium on Applications of Methods of Functional Analysis to Problems of Mechanics held in Marseille, France, on September 1975.

The Proceedings have already been published by Springer Verlag. The complete reference is :

"Applications of Methods of Functional Analysis to Problems in Mechanics" Edited by P. Germain and B. Nayroles; Lecture Notes in Mathematics, no. 503 , Springer-Verlag, Berlin - Heidelberg - New York, 1976.

In the paper acknowledgment to AFOSR sponsorship was incorrectly stated as pertaining to Grant No. AFOSR 74-2704. The correct statement should have been Grant No. AFOSR-76-2889.

2.- L.G. NAPOLITANO :"The functional inverse approach for the variational formulation of boundary value problems".

This paper was presented at the Congress on Modern Problems in Continuum Mechanics held in Turin (Italy) on November 1975.

The proceedings of this Meeting are in press.

3.- L.G. NAPOLITANO :"Functional Analysis Derivation of Hybrid Variational Functionals for Fourth Order Elliptical Operators".

This paper was presented at the Second International Symposium on Finite Elements Methods in Flow Problems held in S. Margherita Ligure (Italy) on June 1976.

The Preprints of the Symposium have been published already and copy of them can be obtained from ICCAD (International Centre for Computer Aided Design) Genova (Italy).

4.- L.G. NAPOLITANO and V. LOSITO :"The Closed Spline Functions".

This paper will be submitted for publication to an International Journal.

Since papers No. 1 and 3 above are readily available (and an appropriate number of copies have already been forwarded to the Sponsoring Agency) the subject Final Report will contain, in extenso, only the papers No. 2 and 4 above which constitute Parts I and II respectively of the Report itself.

PART I

THE FUNCTIONAL INVERSE APPROACH FOR THE VARIATIONAL FORMULATION OF BOUNDARY VALUE PROBLEMS

by

L. G. Napolitano

THE FUNCTIONAL INVERSE APPROACH FOR THE VARIATIONAL FORMULATION OF BOUNDARY VALUE PROBLEMS (°)

by

L.G. Napolitano

1. INTRODUCTION

The present author has developed a functional approach for the unified derivation of hybrid and/or classical variational formulations of boundary value problems [1], [2], [3][4].

This approach, referred to as the " direct approach", considers as given a formally self-adjoint linear boundary value problem and derives its different variational formulation from the stationary properties of a two-field functional defined over two suitably defined linear varieties. The direct approach hinges on an assumed factorization of the differential operator defining the boundary value problem and, ultimately, on the corresponding Gauss-formula.

A more powerful and general approach, to be referred to as "inverse approach", will be presented here.

In dealing with a general variational theory of boundary value problems Lions and Magenes [5] have already pointed out the connection existing, via an underlying Gauss formula, between the classes of problems that can be analysed and the assumed definition of an inner product.

The contention here is that, ultimately and more generally, the connection

(°) Paper presented at the Congress on Modern Problems in Continuum Mechanics, Torino, 1976

is to be established between an abstract Gauss formula and classes of boundary value problems. The inverse approach both accomplishes these tasks and duely exploits its results.

The starting point is an abstract Gauss formula and the end results are variational formulations (derived by the method developed in the direct approach) of the classes of boundary value problems that can be "associated" to the given Gauss formula.

As it will be shown, an abstract Gauss formula involves essentially two Hilbert spaces of elements v and τ defined over a domain Ω, a differential operator G such that $\{G v\}$ is a subset of $\{\tau\}$, a number of boundary spaces defined over partitions of the boundary of Ω and a set of principal and boundary operators. The "degrees of freedom" it affords thus pertain not only to the choice of the definition of inner products but also to the very same choices of the elements v, τ, of the operator G and of their tensorial orders.

The power and generality of the inverse approach stems from the unified derivation of variational formulations for the wide classes of boundary value problems resulting from such diversified choices.

The inverse approach presented in this paper is a natural outcome of the direct approach previously developed by the author. It thus makes uses of many results already established in previous works to which reference is made for greater details and/or for pertinent discussion of the significance of the main assumption which underlies its development.

2.- FUNDAMENTALS OF THE INVERSE APPROACH

2.1 - The abstract Gauss formula

Let Ω be an open bounded subset of R^n with sufficiently smooth boundary $\partial\Omega$ and let ~~there~~ $\partial_{p_j}\cdot\Omega$, $\partial_{n_j}\cdot\Omega$ $(1 \leq j \leq m)$ be $(m > 1)$ arbitrary partitions, not necessarily all distinct, of $\partial\Omega$.

Let $V(\Omega) \subset K(\Omega)$ be two appropriate subsets of the real Hilbert space $L^2(\Omega)$, where the real function $v \in V$ can be of any tensorial order, and G be a differential operator of order m and arbitrary tensorial order. Assume that there is a Gauss's formula of the type :

$$< Gv, \tau > = (v, G^*\tau) - \sum_{j=1}^{m} \ell_j (P_j v, N_j \tau) \qquad (2.1)$$

where $Gv, \tau \subset U_1(\Omega)$ a subset of the $L^2(\Omega)$ space appropriate to their common tensorial order; $<\ ,\ >$ and $(\ ,\ )$ denote inner products in U_1 and V, respectively, G^* is the formal adjoint of G and the ℓ_j's are bilinear symmetrical forms on $\partial\Omega$ which characterize the set of (m) "principal" (resp. "natural") boundary differential operators P_j (resp. N_j) of order (j - 1) [resp. m - j] .

The conditions on $\partial\Omega$ and its partitions necessary for the (formal) validity of the Gauss'formula are assumed to hold [5].

The nature of the elements v and τ (and thus of the operator G) and the inner products $<\ ,\ >$ and $(\ ,\ )$ [and thus of the sets of boundary differential operators] are left "unspecified".

Denoting by ℓ_{p_j} and ℓ_{n_j} the restrictions of ℓ_j to $\partial_{p_j}\cdot\Omega$ and $\partial_{n_j}\cdot\Omega$, respectively, and let :

$$p_j = P_j v \Big|_{\partial_{p_j}\cdot\Omega} \quad ; \quad n_j = N_j G v \Big|_{\partial_{n_j}\cdot\Omega}$$

$$\sum_{j=1}^{m} \ell_{p_j}(P_j \cdot v, N_j \cdot \tau) = \sum_{j=1}^{m} \ell_{p_j}(p_j, N_j \tau) = \ell_p(p, N\tau)$$

$$\sum_{j=1}^{m} \ell_{n_j}(P_j \cdot v, N_j \cdot \tau) = \sum_{j=1}^{m} \ell_{n_j}(P_j \cdot v, n_j) = \ell_n(Pv, n)$$

where :

$$p = [p_1, p_2, \text{-----}, p_m]$$

$$n = [n_1, n_2, \text{---}, n_m]$$

Then the Gauss formula (2.1) can be rewritten as :

$$\langle Gv, \tau \rangle = (v, G^*\tau) - \ell_p(p, N\tau) - \ell_n(Pv, n) \quad (2.2)$$

The thus fas arbitrary element $\tau \in \mathcal{U}_1(\Omega)$ is now restricted to the subset defined by

$$\mathcal{U}_2 = \{\tau \in \mathcal{U}_1(\Omega) \,|\, \tau = Gv + \sigma\, ;\, \sigma \in \Sigma\} \quad (2.3)$$

$$\Sigma = \{\sigma \in \mathcal{U}_1(\Omega) \,|\, G^*\sigma = 0, N\sigma|_{\partial\Omega} = 0\}$$

Gv and σ will be referred to, respectively, ~~to~~ as the compatible (or constitutive) and non-compatible parts of τ.

The subset $\Sigma \subset \mathcal{U}_1(\Omega)$, supposed non-empty [See [4] for relevant examples] is orthogonal to the subset $\{Gv\}$, orthogonality being understood with respect to the inner product $\langle\ ,\ \rangle$.

On account of this, for $\tau, \tau' \in \mathcal{U}_2$ the Gauss formula (2.2) yields, with $G^*Gv = s \in K'(\Omega)$, the dual of $K(\Omega)$:

$$\langle \tau, \tau' \rangle = \langle \sigma, \sigma' \rangle + (v', s) - \ell_p(p', N\tau) - \ell_n(Pv', n) \quad (2.4)$$

In principle, the inner product $\langle \sigma, \sigma' \rangle$ could be further decomposed, by relating it to the inner product pertaining to still another space and to other bilinear forms. This development is not to be pursued here.

It will only be supposed that there is an element r, belonging to the cartesian product space $M_r \subset L^2(\Omega) \otimes L^2(\partial\Omega)$, appropriate to the tensorial order of σ, which determines $\sigma \in \Sigma$ via a pertinent boundary value problem.

The elements s, p, n and r will be interpreted as "generalized" constraints imposed on the element τ of $\mathcal{U}_2$ and it will be assumed that a given set of values for the generalized constraints characterizes a unique element $\tau \in \mathcal{U}_2(\Omega)$ as formalized in the next paragraphs.

2.2 Constraint and solutions spaces.

Define :

$$c = [s, p, n, r] \quad (2.5)$$

as a "constraint element". Its components will be referred to, respectively, as source, principal boundary, natural boundary and constitutive (or compatibility) constraints. Thus they prescribe, in the order, the source (s) in Ω of an element τ, its (m) principal boundary values p_j $(1 \leq j \leq m)$ on the (m) non necessarily coincident subsets $\partial_{p_j}\Omega$ of $\partial\Omega$; its (m) natural boundary values n_j on the (m) complementary subsets $\partial_{n_j}\Omega$, and, finally, the value (r) characterizing its "non-compatible" part σ.

It is assumed that there exist suitable spaces D_s, B_p, B_n and H_t, closed with respect to appropriate norms, such that if

$$\mathcal{C} = D_s \otimes B_p \otimes B_n \otimes H_t$$

then for any $c \in \mathcal{C}$ (i.e. any $s \in D_s,\ p \in B_p,\ n \in B_n,\ t \in H_t$) the linear mapping $F: \mathcal{C} \to \mathcal{U}_2(\Omega)$ is one to one; continuous and bounded. [3] [4].

The space $\mathcal{C}$ will be called the constraint space and its (closed) F-image $H = F(\mathcal{C}) \subset \mathcal{U}_2(\Omega)$ will be called the "solution space". F is an isomorphism between the constraint space $\mathcal{C}$ and the solution space H.

The hypothesis on F is related to the existence and uniqueness of the solution of the following four linear boundary value problems :

$$1)\quad \left.\begin{aligned} &G^* G v = s \\ &P v\big|_{\partial\Omega} = 0 \end{aligned}\right\} \Rightarrow S v = s$$

$$(2.6)$$

$$2)\quad \left.\begin{aligned} &G^* G v = 0 \\ &P_j v\big|_{\partial_{p_j}\Omega} = p_j \qquad (1 \le j \le m) \\ &N_j G v\big|_{\partial_{n_j}\Omega} = 0 \end{aligned}\right\} \Rightarrow L_p v = p$$

$$\left.\begin{aligned} &3)\quad G^* G v = 0 \\ &\quad P_j \cdot v|_{\partial_{p_j}\Omega} = 0 \qquad (1 \le j \le m) \\ &\quad N_j \cdot G v|_{\partial_{n_j}\Omega} = n_j \end{aligned}\right\} \Rightarrow L_n v = n$$

$$\left.\begin{aligned} &4)\quad R\sigma = \tau_1 \\ &\quad B\sigma|_{\partial\Omega} = \tau_2 \end{aligned}\right\} \Rightarrow R\sigma = \tau, \quad \tau = [\tau_1, \tau_2]$$

the operators S, L_p, L_n, R being appropriate restrictions of F^{-1}. The fourth boundary value problem involves the differential operators R (in Ω), of order m, and B (on $\partial\Omega$) which are left unspecified since they depend on the unspecified nature of τ and G. Notice, furthermore, that upon (2.3) :

$$G^*\tau = G^*\sigma$$

whereas, in general (see [4]) $R\sigma \neq R\tau$ since RG is not necessarily the null operator.

The ranges of the above defined operators are the following subsets of the constraint space :

$$\begin{aligned} \mathcal{R}(S) &= \{c_s = [s, 0, n(s), 0]\} = C_s \subset \mathcal{C} \\ \mathcal{R}(L_p) &= \{c_p = [0, p, 0, 0]\} = C_p \subset \mathcal{C} \\ \mathcal{R}(L_n) &= \{c_n = [0, 0, n, 0]\} = C_n \subset \mathcal{C} \\ \mathcal{R}(R) &= \{c_\tau = [0, 0, 0, \tau]\} = C_\tau \subset \mathcal{C} \end{aligned} \qquad (2.7)$$

when accounting for the fact that $F\tau = 0 \iff \tau = 0$.

The inner product among elements of each one of the four basic subspaces reduces to only one of the four contributions contained in the Gauss formula. Thus, alternatively and equivalently, the basic subspaces could have been introduced directly by imposing such a requirement of the Gauss'formula.

2.3 The associated boundary value problem for compatible elements.

The general set up established in the previous paragraphs leads to a) the definition of a boundary value problems for compatible elements, associated with the given Gauss'formula; b) the possibility of deducing classes of variational formulations of this boundary value problem.

By definition, a compatible element $\bar{\varepsilon} = F\bar{c}$ is uniquely characterized by a constraint element $\bar{c}$ of the type :

$$\bar{c} = [\bar{s}, \bar{p}, \bar{n}, 0] \qquad (2.10)$$

The element $\bar{\varepsilon}$ can thus be interpreted as the (generalized) solution of the following boundary value problem :

$$\begin{aligned} &G^* G \bar{v} = \bar{s} \\ &P_j \bar{v}|_{\partial_{p_j}\Omega} = \bar{p}_j \qquad (1 \le j \le m) \\ &N_j G \bar{v}|_{\partial_{n_j}\Omega} = \bar{n}_j \end{aligned} \qquad (2.11)$$

where $\{C\}$ denotes the set generated by **C**. The notation n(s) is meant to indicate that for problem (1) the values of $N_j \cdot G v$ on $\partial_{n_j} \cdot \Omega$ are uniquely determined once s is given.

The corresponding domains are (closed) subspaces of the solution space referred to as basic subspaces and denoted by :

$$X_s = F(C_s);\ X_p = F(C_p);\ X_n = F(C_n);\ X_r = F(C_r) \quad (2.8)$$

The isomorphism F makes it possible to define a pseudo-dimensionality (psd) for subspaces of H as the number of constraints (i.e. number of components of the constraint element) which can be given, independently, arbitrary values. Thus psd H = 4, any of the basic subspaces has psd = 1 and any fixed element $\bar{\tau} \in H$ has psd = 0[2].

The solution space H has the direct sum decomposition ;

$$H = X_s \oplus X_p \oplus X_n \oplus X_r$$

the basic subspaces being mutually orthogonal.

Given the property of H, the only thing which needs to be proven is the orthogonality of the basic subspaces. This follows readily from the definitions (2.7,8) of the subspaces and from the Gauss formula(2.4) rewritten as to exhibit explicitely the components of the constraint element (2.5):

$$\langle \tau, \tau' \rangle = \langle F\tau, F\tau' \rangle + \begin{cases} (v, s') - \ell_p(p, N\tau') - \ell_n(Pv, n') \\ (v', s) - \ell_p(p', N\tau) - \ell_n(Pv', n) \end{cases} \quad (2.9)$$

The element $\bar{\tau}$ of the vector space H will be referred to as the solution vector and the values $\bar{s}, \bar{p}, \bar{n}$ of the source, principal and natural boundary constraints will be referred to as the constraints "of" the solution vector $\bar{\tau}$.

2.4 Linear varieties and two-field variational functional -

Let $\mathcal{Y}_\alpha, \mathcal{Y}_\beta$ be two arbitrary closed subspaces of H; $\bar{\mathcal{Y}}_\alpha, \bar{\mathcal{Y}}_\beta$ their orthogonal complements and $\mathcal{P}_\alpha, \mathcal{P}_\beta, \bar{\mathcal{P}}_\alpha, \bar{\mathcal{P}}_\beta$ the corresponding projection operators (e.g. $\mathcal{Y}_\alpha = \mathcal{P}_\alpha H$; $\bar{\mathcal{Y}}_\alpha = \bar{\mathcal{P}}_\alpha H$).

Denote by $\mathcal{V}_\alpha, \mathcal{V}_\beta$ the linear varieties obtained by translating $\mathcal{Y}_\alpha$ and $\mathcal{Y}_\beta$ by $\bar{\tau}$:

$$\mathcal{V}_\alpha = \{ \tau_\alpha \in H \,/\, \tau_\alpha = \bar{\tau} + y_\alpha \; ; \; y_\alpha \in \mathcal{Y}_\alpha \}$$

$$\mathcal{V}_\beta = \{ \tau_\beta \in H \,/\, \tau_\beta = \bar{\tau} + y_\beta \; , \; y_\beta \in \mathcal{Y}_\beta \}$$

Since $\bar{\mathcal{P}}_\alpha \mathcal{V}_\alpha = \bar{\mathcal{P}}_\alpha \bar{\tau}$ the projection on $\bar{\mathcal{Y}}_\alpha$ of any element $\tau_\alpha \in \mathcal{V}_\alpha$ satisfies the same constraints as the corresponding projection of the solution vector $\bar{\tau}$. Similarly for the elements $\tau_\beta \in \mathcal{V}_\beta$. The elements of the varieties can thus be loosely characterized as satisfying a certain set (in particular null if $\mathcal{V}$=H) of constraints of the solution. [2], [4]

~~Define the two-field variational functional :~~

Define the two-field variational functional :

$$\begin{cases} 2K(\tau_\alpha, \tau_\beta) = \langle \tau_\alpha - \tau_\beta \rangle^2 - \langle \bar{\tau} - \tau_\beta \rangle^2 + \langle \bar{\tau} \rangle^2 = \langle \tau_\alpha \rangle^2 - 2\langle \tau_\alpha - \bar{\tau}, \tau_\beta \rangle \\ \tau_\alpha \in \mathcal{V}_\alpha \ ; \ \tau_\beta \in \mathcal{V}_\beta \end{cases} \qquad (2.12)$$

As proven in [3] this functional attains a stationary value for $\tau_\alpha = \bar{\tau}_\alpha$ and $\tau_\beta = \bar{\tau}_\beta$ such that :

$$\bar{\tau}_\alpha = \bar{\tau} \ ; \quad \bar{\tau}_\beta \in \mathcal{V}_\gamma = \bar{\tau} + \bar{\mathcal{Y}}_\alpha \cap \mathcal{Y}_\beta \qquad (2.13)$$

This functional will be used to obtain, via the Gauss formula, different classes of variational problems equivalent to the boundary value problem (2.11).

3.- VARIATIONAL FUNCTIONALS

The Gauss's formula (2.9) is used to express the two-field variational functional (2.12) in terms of the components of the constraint vector **C**.

By recalling that $\bar{\tau} = G\bar{v} = F[\bar{s}, \bar{p}, \bar{n}, 0]$
and by letting :

$$\tau_\alpha = G v_\alpha + \sigma_\alpha = F c_\alpha = F[s_\alpha, p_\alpha, n_\alpha, t_\alpha]$$

with a similar notation for τ_β, it is readily shown that :

$$K(\tau_\alpha, \tau_\beta) = \frac{1}{2}\langle \tau_\alpha \rangle^2 - \langle \tau_\beta, \tau_\alpha \rangle +$$

$$+ \begin{cases} (v_\beta, \bar{s}) - \underline{\ell_p(p_\beta, N\bar{\tau})} - \ell_n(P v_\beta, \bar{n}) & (3.1) \\ \\ \underline{(\bar{v}, s_\beta)} - \ell_p(\bar{p}, N\tau_\beta) - \underline{\ell_n(P\bar{v}, n_\beta)} & (3.2) \end{cases}$$

and, by further developing $<\tau_\beta, \tau_\alpha>$:

$$K(\tau_\alpha, \tau_\beta) = \frac{1}{2} < \tau_\alpha >^2 - < \sigma_\beta, \sigma_\alpha > +$$

$$+ \begin{cases} (v_\beta, \bar{s} - s_\alpha) - \underline{\ell_p[p_\beta, N(\bar{\tau} - \tau_\alpha)]} - \ell_n[P v_\beta, \bar{n} - n_\alpha] & (3.3) \\ \underline{(\bar{v} - v_\alpha, s_\beta)} - \ell_p[\bar{p} - p_\alpha, N\bar{\tau}_\beta] - \underline{\ell_n[P(\bar{v} - v_\alpha), n_\beta]} & (3.4) \end{cases}$$

Each one of the above four expressions leads to possible forms of one or two-field variational functionals which include, as it will be seen shortly, the classical primal and dual one field functionals. In each case there is a minimum number of requirements to be imposed on the varieties $\mathcal{V}_\alpha$ and $\mathcal{V}_\beta$ in order to eliminate, as obviously needed, the terms containing unknown quantities related to the solution element [i.e. quantities other than the components of the given constraint element $\bar{c}$]. Such terms, underlined in equations (3.1) ÷ (3.4) can be eliminated only by acting on the component of the constraint elements c_α and c_β defining the linear varieties.

Since $\mathcal{V}_\alpha$ and $\mathcal{V}_\beta$ are subspaces translated by $\bar{c}$, the components of c_α and/or c_β to be acted upon must be set equal to the corresponding components of $\bar{c}$. This automatically identifies also the subspaces $\mathcal{Y}_\alpha$, $\mathcal{Y}_\beta$ and, upon equation (2.13), the variety $\mathcal{V}_\gamma$ characterizing the eventual arbitrariness in $\bar{\tau}_\beta$. Other requirements can obviously be added to further restrict the varieties $\mathcal{V}_\alpha$ and $\mathcal{V}_\beta$.

In equation (3.1) the minimum requirement criterion is satisfied if one takes $p_\beta = \bar{p}$ for then the ℓ_p contribution becomes constant and thus irrelevant. Hence $\mathcal{V}_\alpha = H$; $\mathcal{Y}_\alpha = H$ and (with obvious notations) :

$$\mathcal{V}_\beta = F\{[s_\beta, \bar{p}, n_\beta, t_\beta]\}$$

$$X_\beta = F\{[s_\beta, \bar{p}, n_\beta, t_\beta]\} - F[\bar{s}, \bar{p}, \bar{n}, o] =$$

$$= F\{[s, 0, n, t]\} = X_s \oplus X_n \oplus X_t$$

Thus : $\bar{\mathcal{Y}}_\alpha \cap \mathcal{Y}_\beta = \phi$; $\bar{\tau}_\beta = \bar{\tau}$, no indeterminacy in τ_β exists and the functional reads

$$\begin{cases} K = \frac{1}{2} \langle \tau \rangle^2 - \langle \tau_\beta, \tau \rangle + (v_\beta, \bar{s}) - \ell_n (P v_\beta, \bar{n}) \\ \tau \in H \; ; \; \tau_\beta = G\, v_\beta + \sigma_\beta \; ; \; P_j v_\beta |_{\partial p_j \Omega} = \bar{p}_j \quad (1 \leq j \leq m) \end{cases} \qquad (3.5)$$

This is the most general two-field hybrid functional that can be obtained since τ is an arbitrary (i.e. unconstrained) element of H and v_β is subject only to the principal constraints on $\partial_{p_j} \Omega$.

A similar procedure applied to equation (3.2) yields :

$$s_\beta = \bar{s} \; ; \; n_\beta = \bar{n} \; ; \; \mathcal{V}_\alpha = \mathcal{Y}_\alpha = H$$

$$\mathcal{Y}_\beta = X_p \oplus X_t \; ; \; \mathcal{V}_\beta = \bar{\tau} + X_p \oplus X_t$$

and the variational functional reads :

$$\begin{cases} K = \frac{1}{2} \langle \tau \rangle^2 - \langle \tau_\beta, \tau \rangle - \ell_p (\bar{p}, N \tau_\beta) \\ \tau \in H \; ; \; G^* \tau_\beta = \bar{s} \; ; \; N_j \tau_\beta |_{\partial n_j \Omega} = \bar{n}_j \end{cases} \qquad (3.6)$$

No indeterminacy in $\bar{\tau}_\beta$ exists. One of the elements is still completely unconstrained, the other satisfies the source and the natural boundary constraints only.

When considering the other two general expressions for K (τ_α, τ_β) the term $<\sigma_\beta, \tau_\alpha>$ may be made to disappear since otherwise one essentially falls back into the cases already dealt with.

Accordingly, the minimum criterion applied to equation (3. 3) leads to :

$$\sigma_\beta = 0 \qquad P_\beta = \bar{P}$$
$$v_\alpha = H = \mathcal{Y}_\alpha \qquad , \; v_\beta = \bar{\varepsilon} + X_s \oplus X_n$$

and the variational functional reads :

$$\left\{\begin{array}{l} K = \frac{1}{2} < \tau_\alpha >^2 + (v_\beta, \bar{s} - s_\alpha) + \ell_p(\bar{p}, N\tau_\alpha) - \ell_n(P v_\beta, \bar{n} - n_\alpha) \\ P_j v_\beta |_{\partial_{P_j}\Omega} = \bar{P}_j \qquad (1 \le j \le m) \end{array}\right. \qquad (3.7)$$

This is a quite general two-fields hybrid functional since, once again, no constraints are imposes on τ_α and only the principal boundary constraints are imposed on v_β. Compared with the functional (3.5), σ_β is disappeared but the source constraints of τ_α appears explicitely. If the restriction $s_\alpha = \bar{s}$ is imposed on τ_α , the term (,) disappears and :

$$v_\alpha = \bar{\varepsilon} + X_p \oplus X_n \oplus X_\tau \qquad , \quad \{\bar{\tau}_\beta\} = \bar{\tau} + X_s$$

The indeterminacy in $\bar{\tau}_\beta$ is however irrelevant since as said, K would not contain any volume integral of τ_β . The element τ_β can be made to disappear all together by further imposing the natural boundary constraint to the elements τ_α . The single field functional thus obtained is nothing but the classical "dual" variational functional K_D :

$$\left\{\begin{array}{l} K_D = \frac{1}{2} < \tau_\alpha >^2 + \ell_p(\bar{p}, N\tau_\alpha) \\ G^* \tau_\alpha = \bar{s} \quad ; \quad N_j \tau_\alpha \Big|_{\partial n_j \Omega} = \bar{n}_j \qquad (1 \le j \le m) \end{array}\right. \qquad (3.8)$$

From the last expression (3.4) one finally gets :

$$\sigma_\alpha = 0 \quad ; \quad s_\beta = \bar{s} \quad ; \quad n_\beta = \bar{n}$$

$$v_\alpha = X_\alpha \qquad ; \quad v_\beta = \bar{\tau} + X_p \oplus X_\tau$$

$$\{ \tau_\beta \} = \bar{\tau} + X_p \oplus X_\tau$$

and the variational functional reads :

$$\begin{cases} K = \frac{1}{2} \langle G v_\alpha \rangle^2 - (v_\alpha, \bar{s}) - \ell_p [\bar{p} - P_\alpha, N \tau_\beta] + \ell_n [P v_n, \bar{n}] \\ N_j \tau_\beta |_{\Gamma_{n_j} \cdot \Omega} = \bar{n}_j \quad (1 \leq j \leq m) \end{cases} \qquad (3.9)$$

No constraint is imposed on v_α and the indeterminacy in τ_β is once again irrelevant. As before, τ_β can be made to disappear by imposing the principal boundary constraints on v_α . The single field functional thus obtained is nothing ~~both~~ but the classical "primal" variational functional K_p for the boundary value problem (2,11) :

$$\begin{cases} K_p = \frac{1}{2} \langle G v_\alpha \rangle^2 - (v_\alpha, \bar{s}) + \ell_n (P v_\alpha, \bar{n}) = K_p(v_\alpha) \\ P_j v_\alpha |_{\Gamma_{p_j} \cdot \Omega} = \bar{p}_j \quad (1 \leq j \leq m) \end{cases} \qquad (3.10)$$

4. GENERALIZATIONS

The classes of variational functionals can be ~~further~~ widened by further decomposing the subspaces X_p and X_n. The argument will be developed in detail only for X_p. The constraint element corresponding to X_p is :

$$c_p = [0, p, 0, 0]$$

Let J be the index set $[1,2,\ldots,m]$ J_a and J_b any partition of J and put p = p_a + p_b with $p_a = [p_j]$, $j \in J_a$ and $p_b = [p_j]$; $j \in J_b$.

The restriction of < , > to X_p will be written as :

$$\sum_{j\in J_a} \ell_{p_j}(p_j, N_j \tau') + \sum_{j \in J_b} \ell_{p_j}(p_j, N_j \tau') = \tag{4.1}$$
$$= \ell_{p_a}(p_a, N_a \tau') + \ell_{p_b}(p_b, N_b \tau')$$

Each contribution on the right hand side can be interpreted as the restriction of the inner product to two orthogonal complementary subspaces X_{p_a} and X_{p_b} defined by :

$$X_{p_a} = \left\{ \tau \in X_p \,\middle|\, P_j \tau|_{\partial_{p_j}\Omega} = p_j, \ j \in J_a;\ P_j \tau|_{\partial_{p_j}\Omega} = 0, \ j \in J_b \right\}$$
$$X_{p_b} = \left\{ \tau \in X_p \,\middle|\, N_j G \tau|_{\partial_{p_j}\Omega} = 0, \ j \in J_a;\ P_j \tau|_{\partial_{p_j}\Omega} = p_j, \ j \in J_b \right\} \tag{4.2}$$

Indeed, the corresponding constraint elements read :

$$c_{p_a} = [0, p_a + 0, 0, 0]$$
$$c_{p_b} = [0, P_a(p_b) + p_b, 0, 0]$$

so that $c_p = c_{p_a} + c_{p_b}, \forall c_p \in C_p$, the decomposition being unique when the usual assumption upon the existence and uniqueness of the solution of the boundary value problem corresponding to X_{p_b} is made.

Hence :

$$X_p = X_{p_a} \oplus X_{p_b}$$

and the orthogonality follows from equations (4.1) and (4.2) . Notice that for any $\tau \in X_{p_b}$ it is $N_j' \tau \upsilon = 0 \; (j \in J_a)$ on the entire boundary $\partial \Omega$ since for any $\tau \in X_p$ it is $N\tau|_{\partial\Omega} = n = 0$. Either subspace X_{p_a} or X_{p_b} can be further decomposed with the same procedure but such a development will be omitted.

Consider now the functional K (τ_α, τ_β) and weaken the constraints on the variety $\mathcal{V}_\beta$ by letting :

$$\mathcal{V}_\beta = \bar{\mathcal{P}}_\beta [\mathcal{P}_\alpha \bar{\tau} + \bar{\mathcal{P}}_\alpha \tau_0] + \mathcal{Y}_\beta = \tau_1 + \mathcal{Y}_\beta$$

where τ_0 is an arbitrary but fixed element of H.

Hence $\mathcal{Y}_\beta$ is translated by an element τ_1 whose projection on $\bar{\mathcal{Y}}_\alpha$ is no longer equal to that of the solution element but quite arbitrary. It is readily shown that the stationary properties mentioned in paragraph (2.4) still hold with $\mathcal{V}_\gamma$ the translate of $\bar{\mathcal{Y}}_\alpha \cap \mathcal{Y}_\beta$ by τ_1 instead of $\bar{\tau}$.

As a consequence of this new definition of $\mathcal{V}_\beta$ one has "more freedom" in the process of elimination of the underlined terms from equations (3.1) ÷ (3.4). Indeed one can set the components of the constraint element c_β corresponding to the projection $\bar{\mathcal{P}}_\alpha \bar{\mathcal{P}}_\beta$ equal to some fixed but otherwise arbitrary value. As this can be done only when $\bar{\mathcal{P}}_\alpha \neq \phi$, i.e. $\mathcal{Y}_\alpha \neq H$, one readily sees that the suggested procedure amounts to "shift" a constraint from $\mathcal{V}_\beta$ to $\mathcal{V}_\alpha$.

Thus, for instance, one can eliminate the underlined term from equation (3.1) by setting :

$$P_{\beta a} = \bar{P}_a \quad ; \quad P_{\beta b} = P_{bo}$$

with P_{bo} arbitrary but fixed. The variational formulation is equivalent to the original boundary value problem if one restricts τ_α to the subset satisfying the principal boundary conditions $\bar{P}_b$.

Hence the new variational formulation reads :

$$\begin{cases} K = \frac{1}{2} < \tau_\alpha >^2 - < \tau_\beta , \tau_\alpha > + (v_\beta , \bar{s}) - \ell_n (P v_\beta , \bar{n}) \\ P_j v_\alpha \big|_{\partial_{P_j} \Omega} = \bar{P}_j \ , j \in J_b \ ; \ P_j v_\beta \big|_{\partial_{P_j} \Omega} = \bar{P}_j \ , j \in J_\alpha \end{cases}$$

Comparison with equations (3.5) shows that the principal boundary constraint has been split into $(\bar{P}_a + \bar{P}_b$ and the part $\bar{P}_b$ has been "shifted" from the elements τ_β to the elements τ_α .

Similar developments can be applied to the other three expressions (3.2), (3.3), (3.4) and/or for the natural boundary constraints to obtain corresponding new sets of variational functionals.

CONCLUDING REMARKS

The inverse approach developed here makes it possible to unitarely derive hybrid and/or primal and dual variational formulations of classes of boundary value problems by starting from general explicit expressions for Gauss'formulae. The "degrees of freedom" are essentially constituted by the different "physical nature" of the elements τ (e.g. symmetric second order tensors, general second order tensors, higher order tensors, and so on), by the definition of the inner product (which, among other things, characterizes the set of principal and natural boundary differential operators and hence the types of boundary conditions that can be analysed) and by the selection of the "constitutive" part $\sigma\upsilon$ of τ (which, for second and higher order tensors, can be done in more than one way). Changing any one of the above "degrees of freedom" leads to different classes of boundary value problems.

REFERENCES

1.- NAPOLITANO L.G. (1973) :"Unified Functional Approach to the Solution of Formally Adjoint Problems of Continuum Mechanics" - Letters in Applied and Engineering Sciences, vol. 1, pp. 465-479

2.- NAPOLITANO, L.G. (1973) :"Functional Analysis Derivation and Generalization of Hybrid Variational Methods", I.A.R. no. 218 - Final Scientific Report, Grant AFOSR-73-2508

3.- NAPOLITANO, L.G. (1974) :"On the functional Analysis Derivation and Generalization of Hybrid Methods" - Proceedings II Meeting AIMETA, Naples - Also : Meccanica vol. 10, no. 3 Sept. 1975

4.- NAPOLITANO, L.G. (L975) :"Functional Analysis Approach for the Derivation of Hybrid Variational Functionals" - Proc. IUTAM/IUM Simp. Marseille - Also: Application of Methods of Functional Analysis to Problems in Mechanics, Lecture Notes in Mathematics, 503, Springer Verlag, 1976

5.- LIONS, J.L., MAGENES :"Problèmes aux limites non homogènes et applications" - vol. I, Dunod, Paris 1968.

P A R T I I

THE CLOSED SPLINE FUNCTIONS

by

Luigi G.Napolitano - V. Losito

THE CLOSED SPLINE FUNCTIONS

by

Luigi G. Napolitano - V. Losito

1.- INTRODUCTION

The interpolation problem solved by the classical spline functions is well known [1].

Given (n) points (t_i)

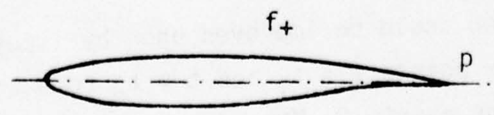
$$a < t_1 < t_2 \ldots\ldots\ldots < t_m < b$$

in the closed interval $[a, b]$ and (n) real numbers (r_i) the interpolating spline-function of order $q \leq n$ corresponding to the (n) points (t_i) is the only function $f \in H^q[a,b]$ which solves the following minimization problem :

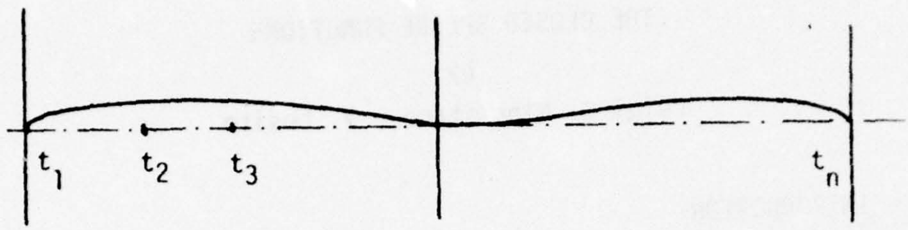
$$\begin{cases} \min\limits_{f \in H^q} \int_a^b \left[f^{(q)}(t) \right]^2 dt & \\ f(t_i) = r_i & 1 \leq i \leq m \end{cases} \qquad (1.1)$$

where $H^q[a,b]$ is the Hilbert space of real functions, defined on $[a, b]$ having a square-integrable q-th derivative. The space H^q quantifies the notion of "degree of smoothness" of the interpolation curve. For q = n the interpolating spline reduces to the unique polynomial of degree (n) passing through the points $f(t_i) = r_i$.

The interpolating spline belongs to the subspace $S \subset H^q$ of real functions s(t)



a) Airfoil



b) Spline-scheme

defined on $[a, b]$ and such that [1] :

a) s is a polynomial of degree (2q - 1) in each of the open intervals
   $(1 \leq i \leq n - 1)$

b) s is a polynomial of degree (q - 1) in $[a, t_1[$ and $]t_n, b]$

c) $s^{(2q - 2)}$ is continuous on $[a , b]$

In many aerospace problems it is necessary to find interpolating functions for "airfoils", i.e. for closed curves. This interpolation problem is often solved by using the classical spline function with $t_i = t_m$ (see figure). Such a procedure is unsatisfactory for two reasons. Firstly, the discontinuities introduced at the airfoil point (usually, the trailing edge) corresponding to the values t_1 and t_m are not those proper of the airfoil (if any). Indeed the classical spline of order q yields different and uniquely defined values for the first (q - 1) derivatives at the points $t = t_1$ and $t = t_n$.

Secondly, upon the property b) of the spline functions, all derivatives from order q to 2 (q - 1) of the interpolating function will vanish at the point P. This expecially for low values of q, much to restrictive in practical applications.

In principle, this approach could be improved upon by imposing a number of additional requirements at the points $t = t_1$ and $t = t_n$ to better approximate the shape of the airfoil at the points P. Not much could however be said about the properties (such as existence, uniqueness, extremality, and so on) of the resulting classes of interpolating functions.

A completely different approach is advocated here. Rather than first assuming that the points t_i belong to an interval $[a,b]$ and then, somehow accounting for the fact that both $t=t_1$ and $t=t_n$ correspond to a same point of the curve to be approximated, the points (t_i) are right from the beginning supposed to belong to an (arbitrary) closed contour C.

By thus doing each t_i corresponds to one and only one point of the curve to be approximated and the interpolating problem for closed curves is consequently cost into its more natural formulation.

In the present paper it will be shown that the interpolation problem thus formulated is amenable to rigorous analysis and leads to a new class of spline-functions defined on a closed contour C.

In wanting of a better terminology, they will be referred to as "closed spline-functions" since they yield interpolating functions for closed curves.

Hilbert space theory of spline-functions will be used. Basic results needed are recalled in paragraph (2). Existence, uniqueness and characterization of the closed spline functions are established in paragraph (3).

These results are restated in conventional (i.e. non abstract space) notations in paragraph (4) where additional relevant properties (following from the basic abstract theory) are also given. A final paragraph offers few comments on the other classes of closed-spline functions which can be constructed by the same approach.

2. - THE ABSTRACT SPLINE FUNCTION THEORY -

The needed basic results of the Hilbert space formulation of spline-function theory are summarized here for ready and convenient reference $\{$See [1] for greater details$\}$.

Let X, Y, Z be three real Hilbert spaces with norms and $T : X \to Y$; $A : X \to Z$ two linear continuous operators, which, without any loss of generality are supposed to be onto.

Given any fixed $z \in Z$, define the set, supposed non empty:

$$I_z = \left\{ x \in X \;/\; Ax = z \right\} \tag{2.1}$$

and consider the following minimum problem :

$$\| T\sigma \|_Y = \min_{x \in I_z} \| Tx \|_Y \tag{2.2}$$

The element $\sigma \in I_z \subset X$, if it exists, is called the interpolating spline corresponding to (T, A, z).

The following existence and uniqueness theorem holds. Theorem 1 (Existence).

<u>Theorem 1</u>

"The solution of problem (2.2) exists for any $z \in Z$ iff N(T) + N (A) is closed in X and is unique iff, in addition, N(T) AN(A) = $\{0\}$ ".

Here N(B) denotes the null space of the operator B.

The definition of the space S of spline-functions follows from the characterization stated in :

Theorem 2 :

"Under the hypothesis of Theorem 1, $\sigma \in I_z$ is the interpolating spline corresponding to (T, A, z) iff

$$< T\sigma, Tx >_Y = 0 \qquad \forall\, x \in N(A) = I_0$$

Here $< , >_Y$ denotes the inner product in Y. Similar notation will be adopted for the inner products."].

The space S is the subspace of X defined by :

$$S = \left\{ s \in X \;/\; < Ts, Tx >_Y = 0 \;;\; \forall x \in N(A) \right\} \qquad (2.3)$$

Theorem 1 can be formulated (and generalized) in S as :

Theorem 3

"For any $z \in Z$ there exists a unique element $\sigma \in S$ such that $A(\sigma) = z$".

The extremal properties of interpolating spline are condensed in the following theorem.

Theorem 4

If z is an arbitrary fixed element of Z and σ is the unique element of S such that A (σ) = z then :

a)

$$\|T(\sigma - x)\|_Y = \min_{x \in S} \|T(s - x)\|_Y \tag{2.4}$$

$\forall x \in I_z$

and any other $\tilde{\sigma} \in S$ having this property belongs to the set

$$\{\tilde{\sigma}\} = \sigma + N(T)$$

b)

$$\|T(\sigma - s)\|_Y = \min_{x \in I_z} \|T(x - s)\|_Y \tag{2.5}$$

and σ is the unique element of I_z having this property.

Further discussion will be limited to the case of more specific interest here, in which the number of interpolating constraints is finite.

The space Z is consequently finite and, upon theorem 1, existence is automatically guaranteed whereas uniqueness requires that N(T) be also finite.

Suppose then that $Z \subset R^n$,with the usual inner product, and that N(T) is of dimension q. Then : a) the operator A can be expressed as :

$$Ax = [\langle K_1, x\rangle_X, \ldots\ldots, \langle K_n, x\rangle_X] \in R^n \tag{2.6}$$

where the K_i's are n independent, linear continuous functional on X; b) the set can be written as :

$$I_z = I_r = \{x \in X \,/\, \langle K_i, x\rangle_X = r_i \;;\; 1 \le i \le n\}; r = (r_1, r_2 \ldots, r_n) \in R^n$$

c) the following characterization theorem holds :

<u>Theorem 5</u> :

" $\sigma \in I_z$ is the interpolating spline-function corresponding to (T, K_i, z) iff there exist (n) coefficients λ_i such that :

$$T'T\sigma = \sum_{i=1}^{n} \lambda_i K_i \quad \in [N(T)]' \tag{2.7}$$

Here T' is the adjoint of T and $[\;]'$ denotes the orthogonal subspace.

The different classes of spline functions that can be obtained from the above abstract formulation depends on the choices of X,Y , T and K_i .

A particular instance is developed in the next paragraph.

3 - The closed spline-functions - Existence, uniqueness and characterization

Let C be a closed contour, sufficiently smooth and regular for the validity of all properties that will be used.

Take $X = H^q(C)$ and $Y = H^\circ(C)$ with their standard inner product.

Denote by τ the curvilinear coordinate along C measured from an arbitrary initial point P_o and normalized with respect to the length ℓ of C.

Thus $0 \leq \tau \leq 1$ and $\tau(P_o) = 0^+ ; \tau(P_o) = 1^-$

As operator $T : X = H^q(C) \to Y = H^\circ(C)$ take $d^{(q)}$ where (d) stands for the derivative with respect to τ . Clearly T is linear and continuous.

Finally, for $z \subset R^m$ the (n) functionals $K_i : H^q(C) \to \mathcal{R}$ be defined by :

$$(K_i, x)_{H^q} = x(\tau_i) = z_i \qquad 1 \leq i \leq m \tag{3.1}$$

with

$$0 = \tau_1 < \tau_2 < \dots\dots < \tau_m < 1$$

The null spaces of the operators A and T are then given by :

$$N(A) = I_0 = \left\{ x \in H^q(C) \,/\, (K_i, x)_{H^q} = x(\tau_i) = 0 ;\ 1 \leq i \leq n \right\}$$

$$N(T) = \left\{ x \in H^q(C) \,/\, x^{(q)} = 0 \right\} \equiv \left\{ x \in H^q(C) \,/\, x = const \right\} \tag{3.2}$$

Hence their dimensions are equal to (n) and to one, respectively, and

$N(T) \cap N(A) = \{0\}$ provided $n > 1$.

The minimum problem (2.2) reads :

$$\int_C \left[\sigma^{(q)}(\tau)\right]^2 d\tau = \min_{x \in I_r} \int_C \left[x^{(q)}(\tau)\right]^2 d\tau$$

$$I_r = \left\{ x \in H^q(C) \,/\, (K_i, x)_{H^q} = x(\tau_i) = r_i \in R ;\ 1 \leq i \leq n \right\} \tag{3.3}$$

$$r = (r_1, \ldots, r_m) \in R^m$$

and, according to theorem 1, it has a unique solution, for any r, as long as $n > 1$.

The existence and uniqueness of $\sigma(\tau)$ having been established, the next basic task is to characterize it.

Upon the definition of adjoint, theorem 5 and equations (2.7), (3.1) one has, subsequently :

$$\langle d^q\sigma, d^q x \rangle_{H^0} = \langle (d^q)' d^q \sigma, x \rangle_{H^q} = \tag{3.4}$$

$$= \langle \sum_{i=1}^{m} \lambda_i K_i, x \rangle_{H^q} = \sum_{i=1}^{m} \lambda_i x(\tau_i) \quad \forall x \in H^q(C)$$

Always upon theorem 5, $\sum_{i=1}^{n} \lambda_i K_i \in [N(T)]^{\perp}$
so that the coefficients λ_i must be such that :

$$< \sum_{i=1}^{n} \lambda_i K_i , x >_{H^q} = 0 \; ; \; \forall x \in N(T)$$

Now, as seen, $\dim N(T) = 1$ so that always upon equation $(3.2)_2$ it suffices to impose that $\sum^{n} \lambda_i K_i$ be H^q orthogonal to unity. On account of eq. (3.1) this leads to

$$\sum_{i=1}^{n} \lambda_i = 0 \tag{3.5}$$

as the only homogeneous condition to be satisfied by the λ_i'. [This is to be contrasted with the situation of classical spline - functions for which the coefficients λ_i must satisfy the (q) homogeneous conditions $\sum_{i=1}^{n} \lambda_i \tau_i^K = 0$ $(0 \leq K \leq q-1)$] .

In order to characterize σ or, equivalently, to obtain the additional conditions which defines the λ_i's uniquely one must express the rightmost term in equation (3.4) as a scalar product in $H^\circ(C)$ valid for any $x \in H^q(C)$.

A Mac-Laurin development of $x(\tau_i)$ with the rest expressed in intergral forms yields :

$$x(\tau_i) = x(0) + \sum_{j=1}^{q-1} \frac{\tau_i^j \, x^{(j)}(0)}{j!} + \oint \frac{(\tau_i - \tau)_+^{q-1}}{(q-1)!} x^{(q)}(\tau) d\tau$$

where

$$(\ell)_+ = \begin{cases} \ell & \text{if } \ell \geq 0 \\ 0 & \text{if } \ell < 0 \end{cases}$$

Thus, on account of eq. (3.5) :

$$\sum_{i=1}^{m} \lambda_i \, x(\tau_i) = \sum_{j=1}^{q-1} \left[\sum_{i=1}^{m} \frac{\lambda_i \tau_i^j}{j!} \right] x^{(j)}(0) + \oint \sum_{i=1}^{m} \frac{\lambda_i (\tau_i - \tau)_+^{q-1}}{(q-1)!} \overset{(q)}{x} d\tau \quad (3.6)$$

Upon the (absolute) continuity of the derivatives $\overset{(q)}{x}(\tau)$ for $1 \le k \le q-1$ one can let :

$$\sum_{j=1}^{q-1} \left[\sum_{i=1}^{m} \frac{\lambda_i \tau_i^j}{j!} \right] x^j(0) = \oint v(\tau) \, x^{(q)}(\tau) \, d\tau$$

$$v(\tau) = \sum_{j=1}^{q-1} \gamma_j \tau^j / j! \quad (3.7)$$

where the (q-1) coefficients γ_j are uniquely determined in terms of λ_i and τ_i . It is convenient to express this functional dependence in terms of the discontinuities of $v(\tau)$ and of its first (q - 1) derivatives at the "initial" point P_0 , given by :

$$\delta v^{(k)} = v^{(k)}(1^-) - v^{(k)}(0^+) = \sum_{j=k+1}^{q-1} \frac{\gamma_j}{(j-k)!} \qquad 0 \le k \le q-1 \quad (3.8)$$

Repeated integrations by parts lead to :

$$\oint v(\tau) \, x^{(q)}(\tau) \, d\tau = \sum_{k=0}^{q-1} (-1)^k \, \delta v^{(k)} \, x^{(q-k-1)}(0) \qquad \forall x \in H^q(C)$$

so that, by comparison with the first of equations (3.7) :

$$(-1)^k \delta v^{(k)} = \sum_{i=1}^{m} \frac{\lambda_i \tau_i^{q-k-1}}{(q-k-1)!} \qquad 0 \le k \le q-2 \quad (3.9)$$

$$\delta v^{(q-1)} = 0$$

The last of these equations is identically satisfied, upon equation (3.8) and the others yield the required (q - 1) equations needed to determine γ_j in terms of λ_i and τ_i. Equations (3.7) and (3.9) affords the required transformation for the first term on the right hand side of eq. (3.6). The transformation of the second term hinges on the identity :

$$(\tau_i - \tau)_+^{q-1} = (-1)^q (\tau - \tau_i)_+ + (\tau_i - \tau)^{q-1}$$

Let, on account of eq. (3.5) :

$$\sum_{i=1}^{m} \lambda_i \frac{(\tau_i - \tau)^{q-1}}{(q-1)!} = (-1)^{q-1} \sum_{j=0}^{q-2} \frac{\alpha_{q+j}}{j!} \tau^j \qquad (3.10)$$

with :

$$\alpha_{q+j} = (-1)^{q-j-1} \sum_{i=1}^{m} \frac{\lambda_i \tau_i^{q-j-1}}{(q-j-1)!} \qquad 0 \leq j \leq q-2 \qquad (3.11)$$

Since :

$$< \alpha, d^q x >_{A^\circ} = 0 \ ; \ \forall x \in H^q(C) ; \forall \alpha = \text{const} \qquad (3.12)$$

The term in α_q does not contribute to the integral appearing in equation (3.6) and one may thus let :

$$\sum_{i=1}^{m} \frac{\lambda_i (\tau_i - \tau)_+^{q-1}}{(q-1)!} = (-1)^q \sum_{i=1}^{m} \frac{\lambda_i (\tau - \tau_i)_+^{q-1}}{(q-1)!} + p(\tau) \qquad (3.13)$$

$$p(\tau) = (-1)^{q-1} \sum_{j=1}^{q-2} \frac{\alpha_{q+j}}{j!} \tau^j$$

Substitution of equations (3.13), (3.7) into equation (3.6) leads finally to :

$$\sum_{i=1}^{m} \lambda_i x(\tau_i) = \langle \psi, d^q x \rangle_{H^0} \quad ; \forall x \in H^q(C) \tag{3.14}$$

$$\psi = (-1)^q \sum_{i=1}^{m} \lambda_i \frac{(\tau-\tau_i)_+^{q-1}}{(q-1)!} + p(\tau) + v(\tau)$$

Comparison between eqs. (3.14) and (3.4) shows , on account of eq. (3.12), that

$$\sigma^{(q)}(\tau) = \psi + \beta_q = (-1)^q \sum_{i=1}^{m} \lambda_i \frac{(\tau-\tau_i)_+^{q-1}}{(q-1)!} + g(\tau) \tag{3.15}$$

$$g(\tau) = \beta_q + p(\tau) + v(\tau) = \sum_{j=0}^{q-1} \frac{\beta_{q+j}}{j!} \tau^j$$

where β_K is an arbitrary constant.

The expression (3. 15) for $\sigma^{(q)}$ contains (n + q) coefficient (n coefficients λ_i and q coefficients β_{q+j}). The following lemma shows that, on account also of equation (3.5), only (n) of them are arbitrary.

Lemma

"The function $\sigma^{(q)}(\tau)$ and its first (q - 2) derivatives are continuous on C".

Proof

Given the properties of polynomials and of the functions $(\tau - \tau_i)_+^{q-1}$ the statement of the lemma needs to be proven only at the point Po.

In the interval $[\tau_1 = 0, \tau_2]$ eq. (3.15) reduces to :

$$\sigma^{(q)}(\tau) = (-1)^q \frac{\lambda_i \tau^{q-1}}{(q-1)!} + \beta_q + p(\tau) + v(\tau) \; ; \; \tau \in [0, \tau_2] \tag{3.16}$$

In the interval $[\tau_m, 1[$ $\quad (\tau - \tau_i)_+ = (\tau - \tau_i)$ for any i so that :

$$\sigma^{(q)}(\tau) = \beta_q + (-1)^q \alpha_q + v(\tau) \; ; \; \tau \in [\tau_m, 1] \tag{3.17}$$

since for $(\tau - \tau_i)_+ = (\tau - \tau_i)$ upon eqs. (3.10) and the second of eqs. (3.13) :

$$(-1)^q \sum_{i=1}^{m} \frac{\lambda_i (\tau - \tau_i)_+^{q-1}}{(q-1)!} = (-1)^q \sum_{j=0}^{q-2} \frac{\alpha_{q+j}}{j!} \tau^j = (-1)^q \alpha_q - p(\tau)$$

From eqs. (3.16) and (3.17) it follows that :

$$\delta\sigma^{(q+k)} = \sigma^{(q+k)}(0^+) - \sigma^{(q+k)}(1^-) = \delta v^{(k)} - p^{(k)}(0) = \delta v^{(k)} + (-1)^q \alpha_{q+k}$$

$$k \in [0, q-2]$$

since, upon the second of eqs. (3.13) $\quad p(0) = 0 \quad$ and

$$p^k(0) = (-1)^{q-1} \alpha_{q+r}$$

Hence, upon equations (3.9) and (3.11) :

$$\sigma^{(q+k)}(1^-) = \sigma^{(q+k)}(0^+) \qquad k \in [0, q-2]$$

and the proposition is proved.

The(q - 1) conditions :

$$\delta\sigma^{(q+k)} = 0 \qquad (0 \leq k \leq q-2)$$

yield the (q - 1) equations which, together with equation (3.5), make it possible to eliminate q coefficients (λ_j , β_{q+j}) out of the (n + q) upon which $\sigma^{(q)}(\tau)$ depends.

The (q - 1) - th derivative of $\sigma^{(q)}$ is constant in each open interval $]\tau_i, \tau_{i+1}[$ $(1 \leq i \leq n)$. Its discontinuity at the points of coordinate τ_i is equal to λ_i.

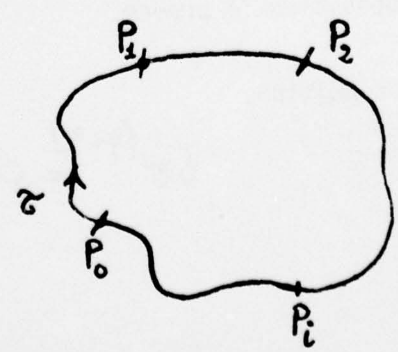
The closed interpolating spline $\sigma(\tau)$ is obtained by integrating equation (3.15) q times. The corresponding (q) arbitrary constants are determined by imposing the continuity of σ and of its first (q - 1) derivatives at P_0. The function thus obtained will contain (n) arbitrary constants which are determined (uniquely, upon theorem 5) from the (n) conditions $\sigma(\tau_i) = r_i$.

Existence, uniqueness and main characterization of the closed spline-functions have thus been proved.

In the next paragraph the additional properties of these classes of functions, as derived from the general theory outlined in paragraph (2), will be presented. To facilitate a more wide spread comprehension abstract space notation will be abandoned. To give a complete panorama, the initial interpolating problem and its solution will also be reformulated in standard notation.

4 - The Closed spline functions - Definition, characterization and properties

Given a sufficiently smooth closed contour C and (n) arbitrary points P_i on it, let τ_i $(1 \leq i \leq n)$ be their curvilinear coordinates (see figure) measured from an arbitrary initial point P_0 and normalized with respect to the length of C.



Define the (n) open intervals $]\tau_i, \tau_{i+1}[$

$$0 = \tau_1 < \tau_2 < \cdots < \tau_n < \tau_{n+1} = 1$$

and prescribe (n) values r_i with

$$(r_1, r_2, \ldots\ldots, r_n) \in \mathbb{R}^n$$

The closed interpolating spline function $\sigma(\tau)$ of degree (q) corresponding to (τ_i, r_i) ($1 \le i \le n$) is defined as the unique element of H^q(C) such that :

$$\int_C \left[\sigma^{(q)}\right]^2 d\tau = \min_{f \in I_r} \int_C \left[f^{(q)}\right]^2 d\tau$$

$$I_r = \left\{ f \in H^q(C) \,/\, f(\tau_i) = r_i \;;\; 1 \le i \le n \right\}$$

In order that $\sigma \in H^q(C)$ be the closed interpolating spline-function corresponding to (τ_i, r_i) it is necessary and sufficient that :

a) σ be a polynomial of degree (2q - 1) in each open interval $]\tau_i, \tau_{i+1}[$ $(\tau_1 = 0 \;;\; \tau_{n+1} = 1)$

b) σ be continuous, on C, together with its first 2(q - 1) derivatives; thus, in particular :

$$\sigma^{(k)}(\tau_i^+) = \sigma^{(k)}(\tau_i^-) \;;\; \forall k \in [0, 2q-2] \;;\; \forall \tau_i$$

c) σ be such that

$$\sigma(\tau_i) = r_i \qquad \forall i \in [1, n]$$

The set of functions $s \in H^q$ satisfying the first two conditions constitute a subspace S (the space of closed spline functions corresponding to τ_i) of dimension n.

For any element $s \in S$ (cfr. theorem 2) :

$$\int_C s^{(q)}(\tau)\, f^{(q)}(\tau)\, d\tau = 0$$

$$\forall f \in I_0 = \left\{ f \in H^q(C) \,/\, f(\tau_i) = 0 \,;\, \forall i \in [1,n] \right\}$$

A base of S is given by (n) s_i's such that $s_i(\tau_j) = \delta_{ij}$ and it is :

$$s = \sum_{i=1}^{n} \lambda_i s_i$$

The following extremal properties hold (cfr. theorem 4) :

Given (n) arbitrary but fixed τ_i , if σ is the unique element of S such that $\sigma(\tau_i) = \tau_i$ then :

A)
$$\int_C \left[\sigma^{(q)}(\tau) - f^{(q)}(\tau)\right]^2 d\tau = \min_{s \in S} \int_C \left[\sigma^{(q)}(\tau) - f^{(q)}(\tau)\right]^2 d\tau$$

and any other element $\tilde{\sigma} \in S$ having this property differs from σ by a constant :

B) for any $s \in S$:

$$\int_C \left[\sigma^{(q)}(\tau) - s^{(q)}(\tau)\right]^2 d\tau = \min_{f \in I_\tau} \int_C \left[f^{(q)}(\tau) - s^{(q)}(\tau)\right]^2 d\tau$$

and σ is the unique element of I_τ having this property.

An element $s \in H^q(C)$ belongs to the space S of the closed spline-functions corresponding to τ_i iff it is representable as :

$$s(\tau) = \sum_{j=0}^{2q-1} \beta_j \frac{\tau^j}{j!} + \sum_{i=1}^{n} (-1)^q \frac{\lambda_i \, (\tau - \tau_i)_+^{2q-1}}{(2q-1)!}$$

where the (n) coefficients λ_i and the (2q) coefficient β_j satisfy the following 2q equations :

$$\sum_{i=1}^{n} \lambda_i = 0$$

$$\delta s^{(k)} = \sum_{j=k+1}^{2q-1} \frac{\beta_j}{(j-k)!} + \sum_{j=k}^{2q-2} \left[(-1)^{q-j-1} \sum_{i=1}^{n} \frac{\lambda_i \, \tau_i^{2q-j-1}}{(2q-j-1)!\,(j-k)!} \right] = 0$$

$$0 \leq k \leq 2q-2$$

where :

$$\delta s^{(k)} = s^{(k)}(1^-) - s^{(k)}(0^+)$$

For any function $f \in H^q(C)$ it is :

$$\int_C s^{(q)}(\tau) f^{(q)}(\tau) \, d\tau = \sum_{i=1}^{n} \lambda_i f(\tau_i)$$

The (n) coefficients (λ_i) represent the values of the discontinuity of the (2 q - 1)-th derivative of s at the points of coordinate τ_i.

When the following (n) equations :

$$s(\tau_i) = r_i \qquad (1 \leq i \leq n)$$

are added to eqs. (4-1), the resulting system of (n + 2q) equations, linear in λ_i and β_j, admits an unique solution.

It may be instructive to compare the closed-spline functions considered herein with the classical spline defined over the interval $[a, b] = [0,1]$ and corresponding to the partition :

$$0 = a = \tau_1 < \tau_2 \cdots\cdots < \tau_n < b = 1$$

In the classical spline-functions the coefficients β_j are identically zero for $q \leq j \leq 2q-1$ and the following (q) relations hold [1] :

$$\sum_{i=1}^{n} \lambda_i \tau_i^k = 0 \qquad 0 \leq k \leq q-1$$

Upon letting :

$$\delta s^{(k)} = s^{(k)}(\tau_n^+) - s^{(k)}(\tau_1^-)$$

one deduces from equations (   ) , (   ) and (   ) that :

$$\bar{\delta} s^{(k)} = \sum_{j=k+1}^{q-1} \frac{\beta_j}{(j-k)!} + \sum_{j=k}^{q-1} \left[(-1)^{q-j-1} \sum_{i=1}^{n} \frac{\lambda_i \tau_i^{2q-j-1} \tau_n^{j-k}}{(j-k)!(2q-j-1)!} \right]$$

$$0 \leq k \leq 2q-2$$

so that, upon noticing that :

$$s^{(k)}(\tau_1^-) = s^{(k)}(0) = \sum_{j=k}^{q-1} \frac{\beta_j}{(j-k)!} \; ; \qquad 0 \leq k \leq 2q-2$$

one finds that :

$$s^{(k)}(\tau_m^+) = s^{(k)}(\tau_1^-) = 0 \qquad q \leq k \leq 2q-2$$

$$\bar{\delta} s^{(k)} \neq 0 \qquad 0 \leq k \leq q-1$$

with the $\bar{\delta} s^{(k)}$ uniquely determined for $0 \leq k \leq q-1$. Thus one recovers that, as said in the introduction if one simply takes $\tau_1 = \tau_m$ and imposes $s(\tau_1) = \tau_1 = \tau_m = s(\tau_m)$ the first (q - 1) derivatives of s (τ) are discontinuous at $\tau_1 = \tau_m$ and the other derivatives, up to the order 2q - 2, vanish identically.

- CONCLUDING REMARKS

As with the classical spline-functions, other classes of closed-spline functions can be introduced by making different assumptions as to the operator A. Thus, for instance, one could construct Hermitian closed-splines [by imposing, as interpolating constraints at the points τ_i , also the values of the first derivative of the function $f \in H^q(C)$] mixed closed-splines constraints equal to linear combinations of values for f and f', Fourier closed-splined, and so on. Furthermore, one may construct similar classes of interpolating and approximating "closed-splines (cfr. [1]).

These further developments will be presented in future reports.

REFERENCES

1.- LAURENT J.P.: "Approximation et Optimization" Hermaun, Paris, 1972